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A STOCHASTIC DESCRIPTION OF PRECIPITATION

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1. Introduction

During the early part of the war years Pierre Massé organized a group for the purpose of studying optimal procedures of development and management of the French hydroelectric and steam power system. The problems encountered by this group included the evaluation of probabilities of excessive discharges, the evaluation of probabilities of excessive droughts, as well as the development of optimal management procedures for the big and small hydroelectric reservoirs.

As the studies of the group progressed, the need for a mathematically tractable description of the random structure of stream flow became more and more imperative. To obtain such a description it was found necessary to start with a description of the random structure of rainfall. The purpose of the present paper is to give a summary account of such a description.

Tentative descriptions of rainfall behavior at one raingage station were introduced around 1944 by M. Loève and independently by E. Halphen. Related formulations have been used more recently by P. A. P. Moran [1] in the study of dams. See also Gani [2] and D. G. Kendall [3]. A description of the areal and temporal behavior of precipitation was introduced by the author around 1947 as an aid in the study of peak discharges. This description, which will be detailed below, does not actually qualify as a stochastic model for precipitation.

A true model should take into account the applicability of the laws of fluid mechanics and thermodynamics. The description given below does not make any provision for the introduction of relations between winds, temperature, origin of air masses, and so forth, and the precipitation itself. Although we have recently attempted to make use of whatever meteorological considerations were accessible to us, at the time of this writing we have not yet met with any reportable success. However there is some hope that our goal will become more attainable in the near future. Also, we hope that the mathematical technique used here will remain applicable in some realistic studies.

Section 2 gives an informal description of the behavior of rainfall that will be converted into formulas in section 5. Sections 3 and 4 introduce the mathemathical apparatus necessary for the conversion.

It is to be noted that the model exposed here is essentially a clustering process

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of the type studied by J. Neyman and E. L. Scott [4] and [5]. The results obtained by these authors are therefore applicable here.

2. An informal description of precipitation

One particular feature of the stochastic structure of rainfall, and a feature which may be of extreme importance in the study of prolonged droughts or the study of the effectiveness of cloud seeding, escaped all our attempts to reasonable description and understanding. We are referring here to the various periodicities, cycles, and effects of external influences which are exposed in the climatological literature. The computations we have been able to perform ourselves failed to give any definite clues on how such long-range fluctuations could be introduced in the model. For this reason all cycles and periodicities, except of course the annual one, will be ignored below. If the existence of other "cycles" or external influences becomes clearly demonstrated, their introduction in the model will probably not present any particular problem. For the sake of completeness let us mention that the possibility of "cycles" may not be rejected outright. For instance, according to some theories, solar activity as measured by sunspot numbers should be correlated with precipitation amounts, the correlation being of opposite signs in the winter months and the summer months. Correlations between sunspot numbers and monthly precipitation at Paris, Helwan, Capetown, Darwin, Wellington, and Apia were computed at the suggestion of E. Halphen on data kindly communicated to us by Miss Frances Clayton. Roughly speaking, the computed correlations do behave according to theory but the effect is too small to lead to a statistically significant result. In spite of positive indications of this nature we shall treat rainfall as a purely random phenomenon subject only to the yearly periodicity.

For simplicity consider a short period such as a week or a month, during which the yearly periodicity may be ignored and during which the process of rainfall production may be regarded as stationary, at least in first approximations. Then the striking feature of rainfall is its excessive areal and temporal variability. Even long steady drizzles such as found in Hawaii show definite and abrupt variations in intensity. The time lag in raingage response does not permit an accurate view of such variations.

Of course, part of the variation detected by sensitive instruments is simply due to atmospheric turbulence in the immediate vicinity of the apparatus. On the other hand "April showers" display this erratic behavior in a manner that cannot be easily ignored. It appears then that at a given location and provided no period shorter than, say, one hour be considered, a scheme of the following type would describe approximately the variations of rainfall intensity. Rainfall occurs only in instantaneous showers. For a given period, the expected number of showers is a fixed quantity A. The actual number of showers and their actual times of occurrence are selected at random according to the familiar Poisson

process. The amount of precipitation in a shower is a random variable independent of the time of occurrence of the shower.

This vastly oversimplified view of the situation already leads to interesting remarks. Assume that the period under consideration is a week. Then, there is a nonzero probability $\exp(-A)$ that no precipitation at all will occur during the week. Second, let $M(a, b) = AF(a, b)$ be the expected number of showers having sizes between a and b. The Laplace transform $f(s) = E \exp[-sX]$ of the total amount X of rainfall occurring during the week may be written

$$\log f(s) = \int_0^\infty [e^{-sx} - 1]\, dM(x) = A \int_0^\infty (e^{-sx} - 1)\, dF(x). \tag{1}$$

Suppose that the mechanism of production of showers is the same for all locations but that the expected number A varies from place to place. Then F remains fixed and f depends only on A. The coefficient of variation of X decreases as $A^{-1/2}$ as A or EX increases. Furthermore, as A increases the probability of clear weather becomes smaller and the distribution of X gets closer to a normal distribution. Such behavior corresponds roughly to observable facts.

To obtain explicit formulas Halphen suggested that F be taken equal to the exponential distribution

$$\frac{dF}{dx} = \frac{1}{a} e^{-x/a}, \qquad x \geqq 0. \tag{2}$$

Then the distribution of X has a mass $\exp(-A)$ at zero and otherwise admits the density

$$p(x) = -iA^2 \exp\left[-\left(A + \frac{x}{a}\right)\right] J_1\left[2i\left(\frac{Ax}{a}\right)^{1/2}\right]\left(\frac{a}{Ax}\right)^{1/2}, \tag{3}$$

where J_1 is the Bessel function of order unity of the first kind.

When a and A vary, the distributions so obtained take various shapes ranging from negative exponential shapes to the usual unimodal shapes. Unfortunately the negative exponential distribution cannot be retained in general. For periods of a week in dry locations the distribution of X is much more skewed than can be allowed by (3). The situation becomes worse if shorter periods are considered, and becomes untenable if the number of showers is taken into account.

For a fixed choice of F the above described process has the feature that rain occurs in sizable instantaneous lumps, so that no drizzle can occur. If, however, A increases indefinitely and M is modified appropriately, something like a continuous drizzle may result. This is the case for instance in Moran's formula, where for $x \geqq 0$

$$\frac{dM}{dx} = \alpha e^{-x} x^{-1}. \tag{4}$$

The resulting distribution for X is then a Gamma distribution. Note that in such a case the probability of rain is unity, for every interval of time, however small.

The most essential defect of the processes just described is that the amounts of precipitation occurring in disjoint periods of time are independent random variables. To remedy this defect one may suppose for instance that the amount of rain per shower is still independent of everything else and has a fixed distribution but that A itself is subject to random fluctuations. More precisely one may suppose that there is a random function A such that, when A has been selected, the expected number of showers in the interval (t_1, t_2) is equal to $\int_{t_1}^{t_2} A(\xi)\, d\xi$. A hump in the function A would then correspond to the passage of a storm, a trough to a period of clear weather. A discretized scheme of this nature was communicated to the author by E. Halphen in the spring of 1946. Note that in Moran's process, variations of A are not feasible since A has become infinite. However, the same effect can be achieved by multiplying the Paul Lévy measure M by a random function. One may even go further and make the measure M itself a random variable, thus changing both the number and the size of the showers.

An extension of such considerations to obtain a description of the areal distribution of rainfall does not present any major difficulty. However, physical considerations, which were notoriously absent in the foregoing, should play a decisive role in the elaboration of an areal model. In the following we have used some words borrowed from the meteorological terminology although we are well aware of the fact that our meteorological friends register amusement, horror, or both, at the sight of such abuse.

The basic element in the construction of an areal model will again be the shower or more exactly the shower cell. By analogy with thunderstorms or "April showers" this may be taken to be a convective cell moving horizontally along a fairly straight path. The direction of the path, the velocity of the cell, the diameter of the cell and its strength can all be considered random, although for simplicity we may for instance want to assume that all cells have the same diameter, say 30 km.

One may assume that, at a given time, the cell distributes its rain uniformly over the area it covers at that time. On the contrary, one may want to assume that the rain distributes itself on this area according to a process which does areawise what Moran's process does timewise.

The next assumption is that cells occur in clusters. Such a cluster could conceivably correspond to a front, the shape of the front determining the shape of the cluster. The number of cells on the front is a random variable. The position of the cells along the front is also determined by some random process. To explain the observed correlations between distant locations it is necessary to assume that the cell clusters move and dissipate after a certain length of time. Furthermore a cell cluster should cross over a given location in a relatively short period of time, a few hours on the average. In such a system, to explain day-to-day observed correlations, it is necessary to assume that the cell clusters them-

selves occur in bunches. A bunch of cell clusters may correspond to what is commonly called a storm. However, the correspondence between the structure just described and natural storms is tenuous and needs further elaboration. Fortunately for applications, in many regions the physical fronts become so confused that the above description may not lead to entirely erroneous conclusions.

If in a "storm" the "fronts" are placed independently of each other, if in a front the "cells" are placed independently of each other and, further, if motions are disregarded, the process just described is plainly a two-stage clustering process in the sense of [4]. It will be shown in section 4 that motions can be allowed for by introducing the time as a supplementary coordinate. Independent positioning of cells may lead to overlap of two or more convective cells. If only geographical position is taken into account overlapping cells may be construed as cells placed at different altitudes. Processes in which overlap of cells or fronts is prevented or controlled can be constructed. Unfortunately such processes are much more difficult to handle. They do not fall naturally in the domain of clustered processes described in the next section.

The influence of geographical structures such as mountain ranges has not been mentioned above. Some *ad hoc* procedures permitting their introduction will be briefly expounded in section 5.

3. Stochastic processes whose index set is a vector space

The stochastic processes used in the construction of the precipitation model to be described here are random set functions of the kind studied in [6]. For the present purposes it appears more convenient to replace the fields or rings of sets used for indices in [6] by linear spaces of numerical or complex-valued functions. To this effect the following definitions will be used.

Suppose a fundamental probability space $\{\Omega, \mathfrak{A}, P\}$ is given. Let V be the space of all $\mathfrak{A}$-measurable numerical functions defined on Ω.

DEFINITION. *A linear stochastic process on F is a linear map $u \to X(u)$ from F to V.*

It is occasionally convenient though hardly necessary in what follows to use complex linear spaces instead of real linear spaces. A linear process would then be a complex linear map from F to the space of complex-valued random variables on $\{\Omega, \mathfrak{A}\}$.

Let F^* be the algebraic dual of the vector space F. For $v \in F^*$ and $u \in F$ let $\langle v, u \rangle$ be the value of v at u. Let $\mathfrak{B}$ denote the smallest σ-field of the subsets of F^* with respect to which for every $u \in F$ the function $v \to \langle v, u \rangle$ is measurable.

A linear stochastic process on F can also be identified to a measurable map from $\{\Omega, \mathfrak{A}\}$ to $\{F^*, \mathfrak{B}\}$. The distribution of the process is the probability measure induced by this map on the σ-field $\mathfrak{B}$. Such a distribution is uniquely

determined by a Fourier transform or characteristic function definable as follows. The characteristic function of the linear process $\{x(u); u \in F\}$ is simply the complex-valued function φ defined on F by the formula

$$\varphi(u) = E\{\exp [iX(u)]\}. \tag{5}$$

A characteristic function possesses the properties

(a) φ is of positive type, that is, for every complex-valued function c defined on F and vanishing everywhere except for a finite subset of F the following inequality holds

$$\sum_{u \in F} \sum_{v \in F} c(u)\bar{c}(v)\varphi(u - v) \geqq 0. \tag{6}$$

(b) The function φ is continuous along the rays of F. In other words for every fixed $u \in F$ the function $\alpha \to \varphi(\alpha u)$ is a continuous function of α.

(c) $\varphi(0) = 1$.

Conversely a slight modification of a theorem of Bochner ([6], theorem 5.44), gives the following result.

PROPOSITION 1. *Let φ be a complex function defined on the real linear space F. If φ satisfies conditions* (a), (b), *and* (c) *there is one and only one probability measure P on $\{F^*, \mathfrak{B}\}$ such that $\varphi(u) = \int \exp \{i\langle x, u\rangle\} P(dx)$ for every $u \in F$.*

From a given linear process one can obtain new linear processes by linear transformations and by completion of F (see [6]).

Let $X = \{X(u); u \in F\}$ be a linear process on F. Let G be another linear space and let T be a linear map from G into F. For each $v \in G$ let $Y(v) = X(Tv)$. Then $\{Y(v); v \in G\}$ is a linear process on G. If φ is the characteristic function of X then $v \to \varphi(Tv)$ is the characteristic function of Y.

Suppose now that F is a topological vector space with topology $\mathfrak{I}$. Let $\tilde{F}$ be the completion of F for the uniform structure induced by $\mathfrak{I}$. If the characteristic function φ of the process X is $\mathfrak{I}$-continuous at the origin of F then the map $u \to X(u)$ from F to the space V of random variables on $\{\Omega, \mathfrak{A}, P\}$ is continuous in probability. To see this, note that if $\{u_\nu\}$ converges to zero in F then $\{\varphi(\alpha u_\nu)\}$ converges to unity uniformly on bounded sets of values of the real variable α. Therefore $\{X(u_\nu)\}$ converges in distribution, hence in probability to zero. The map $u \to X(u)$ being linear is then automatically uniformly continuous. Therefore it can be extended by continuity to the whole of $\tilde{F}$. The extended process is defined only up to a set of probability zero so that, a priori, it may not be a linear process.

In some cases the map $u \to X(u)$ is only sequentially continuous for the topology $\mathfrak{I}$. In such a case it may not be possible to extend X to the whole of $\tilde{F}$ but only to a part of $\tilde{F}$ as follows. If X is sequentially continuous at the origin of F and if $\{u_n\}$ is a Cauchy *sequence* in F then $X(u_n) - X(u_m)$ converges to zero in probability as $\min(m, n)$ tends to infinity. Therefore $\{X(u_n)\}$ is a Cauchy sequence which has a limit in probability in V. Let $\tilde{F}_1$ be the subset of $\tilde{F}$ formed by the limits of Cauchy sequences of F (this is the first Baire hull of

F). It is easily verified that if $\{u_n\}$ and $\{u_n'\}$ are two Cauchy sequences converging to the same $u \in \bar{F}_1$ then the limits of $\{X(u_n)\}$ and $\{X(u_n')\}$ are equivalent.

Whether an extension has been performed to $\bar{F}_1$ or to the whole of $\bar{F}$ one can, by changing the underlying probability space, transform the extension into a linear process on $\bar{F}_1$ or on $\bar{F}$ as the case may be. This follows for instance from proposition 1.

Besides constructions by linear transformations and completion we shall use an operation which in particular cases has received the names "branching," "clustering," or "subordination." To describe it we shall have to open a parenthesis on "processes with independent increments" or "additive processes."

Let S be an arbitrary set. Let $\mathfrak{R}$ be a ring of subsets of S and let F be the vector lattice of numerical functions u of the form

$$u = \sum_j \alpha_j I_{A_j}, \tag{7}$$

where the α_j are real numbers, the summation is taken over a finite set, and where I_{A_j} is the indicator of a set $A_j \in \mathfrak{R}$.

A linear process X defined on F is called a process of additive type if for every finite set $\{u_j; j = 1, 2, \cdots, n\}$ of elements of F, two by two disjointness (that is, $u_j u_k = 0$ for $j \neq k$) implies that the random variables $\{X(u_j); j = 1, 2, \cdots, n\}$ form an independent system.

A process of additive type X will be called decomposable if for every $A \in \mathfrak{R}$ and every $\epsilon > 0$ there is a partition $\{A_j; j = 1, 2, \cdots, n\}$ of A by elements of $\mathfrak{R}$ and real numbers $\{\alpha_j\}$ such that

$$P\{|X(I_{A_j}) - \alpha_j| > \epsilon\} < \epsilon \tag{8}$$

for every $j = 1, 2, \cdots, n$.

If the numbers α_j can be taken equal to zero, the process will be called quasi-centered.

If X is a decomposable process, then for every $A \in \mathfrak{R}$ the distribution of $X(I_A)$ is infinitely divisible. Therefore, there exist numbers $m(A)$, and $\sigma^2(A) \geqq 0$ and a Paul Lévy measure $M(A)$ on the real line deprived of its origin such that

$$\log E\, e^{itX(I_A)} = itm(A) - \frac{t^2}{2}\sigma^2(A) + \int \left[e^{it\xi} - 1 - \frac{it\xi}{1+\xi^2}\right] M(A, d\xi). \tag{9}$$

It follows easily that the three functions m, σ^2, and M are additive functions of the set $A \in \mathfrak{R}$. If $u \in F$ has the form $u = \sum \alpha_j I_{A_j}$ for disjoint sets A_j then

$$\log E\, e^{iX(u)} = \sum_j \left\{ i\alpha_j m(A_j) - \frac{\alpha_j^2}{2}\sigma^2(A_j) + \int \left[e^{i\alpha_j\xi} - 1 - \frac{i\alpha_j\xi}{1+\xi^2}\right] M(A_j, d\xi) \right\} \tag{10}$$

Equivalently, in integral notation and for every $u \in F$ one can write

$$\log E\, e^{iX(u)} = i\int u(s)m(ds) - \frac{1}{2}\int u^2(s)\sigma^2(ds) + \iint \left[e^{iu(s)\xi} - 1 - \frac{iu(s)\xi}{1+\xi^2}\right] M(ds, d\xi). \tag{11}$$

In the last integral, the integration operation is performed first on ξ and then on $s \in S$. However, since only finite sums are involved in the second step, the order of operation is irrelevant.

The subordination operation consists simply in giving a distribution to the triplet of additive set functions $\{m, \sigma^2, M\}$. For the additive set function m this does not present any particular difficulty. For M the following considerations may be helpful. First note that instead of randomizing σ^2 and M one can just as well randomize the Khinchin measure K defined by

$$K = \sigma^2\delta + \frac{\xi^2}{1 + \xi^2} M, \tag{12}$$

where δ is the mass unity at zero on the real line. The advantage of this transformation is that the only requirement to be imposed on K is that K be additive on $\mathfrak{R}$ and that for fixed A the measure $K(A)$ be a bounded positive measure on the real line.

The following properties are special cases of more general results adapted for use in the present circumstances.

PROPOSITION 2. *Let $\mathcal{Y}$ be a vector lattice having a unit element I. Let $\mathfrak{X}$ be the space of all relatively bounded linear functionals on $\mathcal{Y}$ and let $\mathfrak{X}^+$ be the space of positive linear functionals on $\mathcal{Y}$. Let $\mathfrak{B}$ be the σ-field induced by $\mathcal{Y}$ on $\mathfrak{X}^+$.*

In order that a complex-valued function φ defined on $\mathcal{Y}$ be the characteristic function of a probability measure on $\{\mathfrak{X}^+, \mathfrak{B}\}$ it is necessary and sufficient that φ satisfy the requirements

(1) *φ is of positive type, $\varphi(0) = 1$ and φ is continuous on the rays of $\mathcal{Y}$.*

(2) *For every y in the positive cone $\mathcal{Y}^+$ of $\mathcal{Y}$ the function $t \to \varphi(ty)$ of the real variable t is the characteristic function of a measure carried by $R^+ = [0, \infty)$.*

PROOF. The necessity of the requirements is obvious. To prove this sufficiency one may proceed as follows. Let $\mathfrak{F} = \times\{R^+; y \in \mathcal{Y}^+\}$ the topological product of positive parts of real lines corresponding to $\mathcal{Y}^+$. Consider first a countable subset D of $\mathcal{Y}^+$ and the corresponding product $\mathfrak{F}_D = \times\{R^+; y \in D\}$. Assume that D contains the unit of $\mathcal{Y}$. One can identify $\mathfrak{X}^+$ to a subset of $\mathfrak{F}$. Let F_D be the projection of $\mathfrak{X}^+$ on $\mathfrak{F}_D$. The set F_D is closed in $\mathfrak{F}_D$. Indeed let B be the space spanned by D in $\mathcal{Y}$. Every positive linear form on B can be extended to a positive linear form on $\mathcal{Y}$ (since D contains the unit of $\mathcal{Y}$). Therefore F_D can be identified to the space of restrictions to D of positive linear functionals on B. Consequently F_D is closed in $\mathfrak{F}_D$. More precisely, F_D is a subset of $\mathfrak{F}_D$ defined by a family of "closed" relations of the form $\sum \alpha_j f(y_j) = 0$ or $\sum \alpha_j f(y_j) \geqq 0$.

According to Kolmogorov's theorem there exists on $\mathfrak{F}_D$ a probability measure P_D defined on the Borel sets of $\mathfrak{F}_D$ and having on B the required characteristic function. Furthermore P_D is tight [7] on the compacts of $\mathfrak{F}_D$. If S is a subset of $\mathfrak{F}_D$ defined by a finite number of relations of the type just described, then $P_D(S) = 1$. Therefore since P_D is tight and since F_D is the intersection of a decreasingly directed family of closed sets of the type S the measure $P_D(F_D)$ is also equal to unity. It follows from this that in $\mathfrak{F}$ the set $\mathfrak{X}^+$ has outer measure

unity so that the measure P constructed by Kolmogorov's procedure can be restricted to $\{\mathfrak{X}^+, \mathfrak{B}\}$.

Consider now a locally compact space Ξ which is countable at infinity. Let $\mathfrak{K}$ be the space of continuous numerical functions having compact support on Ξ. Let $\mathfrak{R}$ be a ring of subsets of a set S. Assume that there exists a sequence $\{A_n\}$ of elements of $\mathfrak{R}$ such that every $A \in \mathfrak{R}$ is contained in one A_n. Let H be the vector lattice of functions defined on $S \times \Xi$ by finite sums of the type

$$h(s, \xi) = \sum_j u_j(s)v_j(\xi), \tag{13}$$

with $u_j \in F$ and $v_j \in \mathfrak{K}$. Let H^* be the space of linear functionals on H and let Q be the cone of positive elements of H^*. Finally, let $\mathfrak{B}$ be the σ-field induced by H on Q.

THEOREM 1. *Let φ be a complex function defined on H. In order that φ be the characteristic function of a probability measure on $\{Q, \mathfrak{B}\}$ it is necessary and sufficient that φ satisfy the requirements*

(1) *φ is of positive type, continuous on the rays of H and $\varphi(0) = 1$.*

(2) *For every fixed h in the positive cone H^+ of H the function $\alpha \to \varphi(\alpha h)$ is, as function of the real-valued α, the characteristic function of a nonnegative random variable.*

If these conditions are satisfied there exists a uniquely determined probability measure having characteristic function φ which is tight on the $w[H^*, H]$ compact subsets of Q.

PROOF. Let $\{g_n\}$ be an increasing sequence of elements of $\mathfrak{K}^+$ such that $\sup g_n = 1$ and such that g_{n+1} be equal to unity on a neighborhood of the support of g_n. Let h_n be the function defined on $S \times \Xi$ by $h_n = I_{A_n} g_n$. For each integer n let H_n be the space of functions γ of the type $\gamma = h_n u$ for $u \in H$. The map $u \to T_n' u = h_n u$ is a positive linear map of H onto H_n. Let L (resp. L_n) be the space of relatively bounded linear functionals on H (resp. H_n). The operation T_n' has transpose T_n defined by $\langle \mu, T_n' u \rangle = \langle T_n \mu, u \rangle$ for $\mu \in L_n$ and $u \in H$. This transpose T_n is a positive linear map of L_n into L. Since H_n is a subspace of H the function φ satisfies (1) and (2) on H_n. However H_n is a vector lattice having h_n for unit element. Thus proposition 2 is applicable to φ on H_n.

Therefore, there exists a probability measure P_n' which is tight on the $w[L_n, H_n]$ compact subsets of L_n^+ and has on H_n characteristic function φ. The map T_n transforms P_n' into a probability measure $P_n = T_n P_n'$ which is tight on the $w[L, H]$ compact subsets of L^+. The characteristic function φ_n of P_n on H is simply given by

$$\varphi_n(u) = \varphi(T_n' u) = \varphi(h_n u). \tag{14}$$

For each $u \in H$ let $X_n(u)$ be a random variable having characteristic function $\alpha \to \varphi_n(\alpha u) = E\{\exp [\alpha i X_n(u)]\}$.

Let $X(u)$ have characteristic function $\alpha \to \varphi(\alpha u)$. For every $\epsilon > 0$ there exists a number $b(\epsilon, n)$ such that

$$P\{X(h_n) \geqq b(\epsilon, n)\} < \frac{\epsilon}{2^n}. \tag{15}$$

This implies $P\{X(h_m h_n) \geqq b(\epsilon, n)\} < \epsilon/2^n$. Equivalently, $P\{X_m(h_n) \geqq b(\epsilon, n)\} \leqq \epsilon/2^n$. Finally, this implies

$$P\{\sup_n X_m(h_n) \leqq b(\epsilon, n)\} \geqq 1 - \epsilon. \tag{16}$$

Let then $B_n(\epsilon)$ be the subset of L^+ defined by the inequalities $\{\mu: \langle \mu, h_n \rangle \leqq b(\epsilon, n)\}$ and let $B(\epsilon) = \cap_n B_n(\epsilon)$.

The set $B(\epsilon)$ is a $w[L, H]$ compact subset of L^+. The preceding argument shows that $P_m[B(\epsilon)] \geqq 1 - \epsilon$ for every integer m. In other words, the set $\{P_m; m = 1, 2, \cdots\}$ is $w[L, H]$ tight on L^+. It follows that the closure of this set for the usual definition of convergence of probability measures is a compact set. Finally, since φ_m tends to φ as m tends to infinity, the sequence $\{P_m\}$ has a unique cluster point P which is therefore a tight probability measure on L^+. This concludes the proof of the theorem.

The elements of the set Q (equal to L^+) can be described as follows. If $q \in Q$ then for each $A \in \mathfrak{R}$ the function $\gamma \to \langle q, I_A \gamma \rangle$ defined on $\mathfrak{K}$ is a positive linear functional, hence representable as an integral with respect to a positive measure defined on the Borel sets of Ξ and attributing finite measures to relatively compact Borel sets. For a fixed $\gamma \in \mathfrak{K}^+$ the function $A \to \langle q, I_A \gamma \rangle$ is a positive additive set function on the ring $\mathfrak{R}$.

Returning to the construction of processes of additive type, take for Ξ the real line deprived of its origin. A Paul Lévy set function M of the type occurring in formula (9) is an element of Q subject to the restriction that for each $A \in \mathfrak{R}$ the integral

$$\int \frac{\xi^2}{1 + \xi^2} M(A, d\xi) \tag{17}$$

is finite. A characteristic function φ of a probability measure P on $\{Q, \mathfrak{B}\}$ must be subjected to further requirements to insure the validity of (17). To describe these requirements it is convenient to enlarge the domain of definition of φ. Since for $u \in H^+$ the random variable $\langle q, u \rangle$ is a nonnegative random variable, the expectation $E\{\exp [z\langle q, u \rangle]\}$ is well defined (by φ) not only for values of z which are purely imaginary but also for values of $z = \alpha + i\beta$ where β is real and α is negative. Therefore the domain of φ can be extended to all complex functions of the type $w = -u + iv$ with $u \in H^+$ and $v \in H$. The extension so obtained will be called the Laplace transform of P. In fact, this Laplace transform is already well defined by its values on H^+.

For $u \in H^+$ let $f(u) = E\{\exp [-\langle q, u \rangle]\} = \varphi(-iu)$. Let $\{g_n\}$ be the sequence of numerical functions defined on Ξ which was used in the proof of theorem 1. Let γ_n be the function

$$\gamma_n(\xi) = g_n(\xi) \frac{\xi^2}{1 + \xi^2}. \tag{18}$$

If φ is the characteristic function of a random element taking values in Q and satisfying (17), then for each $A \in \mathfrak{R}$ the random variables $\langle q, I_A 2_n \rangle$ must increase to a limit random variable. For this to be true it is necessary that the ordinary Laplace transform $\alpha \to f_n(\alpha) = f(\alpha I_A \gamma_n)$, defined for $\alpha \geqq 0$, converge, as n increases, to the Laplace transform of a probability measure. In other words, the f_n must be equicontinuous at the origin. However, if the condition holds, then for every $u \in H^+$ the Laplace transforms $\alpha \to f(\alpha u \gamma_n)$ are equicontinuous at the origin. Conversely, if such a condition is satisfied, the increasing sequence of random variables $\{\langle q, u\gamma_n \rangle\}$ converges (pointwise) to a limit which is a random variable. Finally, this gives

PROPOSITION 3. *Let Ξ be the real line deprived of its origin. Let φ be a complex function defined on H and satsifying conditions* (1) *and* (2) *of theorem* 1. *Let $\{g_n\}$ be a sequence of elements of $\mathfrak{K}^+$ such that* $\sup g_n = 1$ *and such that g_{n+1} be unity on a neighborhood of the support of g_n. Let γ_n be defined by* (18). *For φ to be the characteristic function of an element of Q satisfying requirement* (17) *it is necessary and sufficient that for every $A \in \mathfrak{R}$ the sequence of characteristic functions $\alpha \to \varphi(\alpha I_A \gamma_n)$ be equicontinuous at $\alpha = 0$.*

Assuming this further requirement satisfied, the domain of φ can be extended to all functions of the form $w = -u + iv$ with $u \geqq 0$ and both u and v of the form $\sum_j p_j(s) r_j(\xi)$ with $p_j \in F$ and r_j continuous on Ξ and bounded by a multiple of $\xi^2/(1 + \xi^2)$. The random variables undergo a corresponding extension. This extension may depend on the particular choice of the sequence $\{g_n\}$ but only by modifications on sets of measure zero.

A formulation of the conditions to be satisfied by a function φ to be the characteristic function of a system $\{m, \sigma^2, M\}$ as occurs in formula (11) presents now no special difficulty and will be left to the care of the reader. Such a function would initially be defined on the product $F \times F \times H$ but its domain can be extended to $F \times F \times \overline{H}$ where $\overline{H}$ is the space of functions of the type $\sum u_j v_j$ with $u_j \in F$ and v_j continuous, bounded by a multiple of $\xi^2/(1 + \xi^2)$. Finally the domain of φ can be extended to cover complex functions, thus giving a Laplace transform f defined for triplets $\{w_1, w_2, w_3\}$ with $w_j = + u_j + iv_j$ and $u_1 = 0$ such that the $(-u_j)$ be positive and such that u_1, u_2, v_1, v_2 belong to F while u_3 and v_3 belong to $\overline{H}$. The domain of f is then $\{iF, F^- \times iF, \overline{H}^- \times i\overline{H}\}$ and f is defined by

$$(19) \qquad f(w_1, w_2, w_3) = E\left\{\sup\left[\int w_1(s) m(ds) + \int w_2(s)\sigma^2(ds) + \iint w_3(s, \xi) M(ds, d\xi)\right]\right\}.$$

Let then f be the Laplace transform of such an $\{m, \sigma^2, M\}$ process. To each choice of m, σ^2, and M corresponds a process of additive type on the ring $\mathfrak{R}$. Its characteristic function is given by (11) when m, σ^2, and M are given. It follows that the subordinated process Z obtained when $\{m, \sigma^2, M\}$ is random has a characteristic function

$$E\, e^{iZ(u)} = f(w_1, w_2, w_3), \tag{20}$$

$$w_1(s) = iu(s), \tag{21}$$

$$w_2(s) = -\frac{1}{2}u^2(s), \tag{22}$$

$$w_3(s, \xi) = e^{i\xi u(s)} - 1 - \frac{i\xi u(s)}{1 + \xi^2}. \tag{23}$$

In the present paper we shall be particularly interested in the case where the decomposable process associated with a triplet $\{m, \sigma^2, M\}$ is itself a positive process. For this to be true it is necessary that $\sigma^2 = 0$ and that for each A the measure $M(A, \cdot)$ be carried by the interval $(0, \infty)$ of the line. Furthermore, m itself must then be positive and M must satisfy the requirement that

$$\int \frac{\xi}{1 + \xi^2} M(A, d\xi) < \infty. \tag{24}$$

That σ^2 must vanish and that M must be carried by the positive part of the line results from the unboundedness of random variables having an infinitely divisible distribution. That $m(A)$ must be positive and that (24) must hold results from the fact that for every $\epsilon > 0$ the random variable whose characteristic function is

$$\exp\left\{\int_\epsilon^\infty (e^{i\alpha\xi} - 1)M(A, d\xi)\right\} \tag{25}$$

has a positive nonzero probability of being equal to zero. Hence, for every $\epsilon > 0$ the function

$$i\alpha\left[m(A) - \int_\epsilon^\infty \frac{\xi}{1 + \xi^2} M(A, d\xi)\right] + \int_0^\epsilon \left[e^{i\alpha\xi} - 1 - \frac{i\alpha\xi}{1 + \xi^2}\right] M(A, d\xi) \tag{26}$$

must be the logarithm of the characteristic function of a nonnegative variable. Letting ϵ tend to zero it follows that

$$m(A) - \int_0^\infty \frac{\xi}{1 + \xi^2} M(A, d\xi) \geqq 0. \tag{27}$$

In such a case formula (11) can be written in the simple form

$$\log E\, e^{iX(u)} = i \int u(s)m(ds) + \iint [e^{iu(s)\xi} - 1]M(ds, d\xi). \tag{28}$$

Formulas (20) to (23) inclusive can be simplified accordingly.

One of the most common examples of subordination is the one in which the random measure M is always concentrated at unity on the line and where m vanishes altogether. The resulting process on $\mathfrak{R}$ can then be roughly described as follows. One chooses at random, according to a prescribed distribution, a positive finitely additive set function M on $\mathfrak{R}$. For each set $A \in \mathfrak{R}$ one chooses a random variable $N(A)$ having a Poisson distribution with expectation $M(A)$. The choice is made in such a way that to disjoint sets correspond independent

variables. One can then select in A exactly $N(A)$ points independently of one another, the probability that a point falls in $B \subset A$ being $M(B)/M(A)$. It is also possible to give each of the selected points a mass selected at random, according to a fixed distribution and independently of everything else.

It is to be noted that the random set functions whose existence is implied by theorem 1 need not be countably additive on the ring $\mathfrak{R}$. So far as this author knows, there is no simple requirement insuring the countable additivity of a random set function. The problem is of the same nature as the problem of existence of regular conditional probabilities and can often be handled in a similar manner.

Suppose for instance that S is a locally compact space which is countable at infinity or that S is a Borel set in a Polish space. In the first instance let K be the space bounded continuous numerical functions having compact support on S. In the second instance let K be a vector lattice of bounded Borel functions such that pointwise convergence to zero of decreasing sequences of elements of K implies their uniform convergence. Suppose in addition that $u \in K$ implies $u \wedge 1 \in K$ and that K induces the σ-field of Borel sets of K. Replacing F by K the arguments of theorem 1 are still directly applicable. The sample values of the resulting random element are then automatically representable as integrals with respect to positive countably additive measures. Further, note that under the assumptions of theorem 1, it is not obvious that separate countable additivity on $\mathfrak{R}$ and $\mathfrak{K}$ would imply countable additivity on H itself. That this is indeed the case can be shown by making use of the locally compact structure of Ξ. The proof will not be given here.

Since countably additive measures are often more convenient to handle than finitely additive set functions it may be desirable to modify the framework used for the description of decomposable processes, and thus the related framework of theorem 1, to insure automatic countable additivity whenever possible. For this purpose, assume that S itself is a locally compact space which is countable at infinity and replace the lattice F used previously by the lattice K of continuous numerical functions with compact support. One could also take for S a Borel set in a Polish space and then take for K a suitable vector lattice of Borel functions.

The space H is then replaced by the space of functions which are finite sums $\sum u_j v_j$ with $u_j \in \mathrm{K}$ and $v_j \in \mathfrak{K}$. A complex-valued function φ satisfying the conditions of theorem 1 is automatically the characteristic function of a random positive linear functional on H. A sample value is automatically representable as an integral with respect to a countably additive measure. This countably additive measure is obtainable by extension of the domain of the linear functional to a space which includes at least all bounded Baire functions having compact support on $S \times \Xi$. The extension is uniquely defined by the properties of linearity, positivity, and countable additivity.

Whenever random measures on a set S will be considered in the sequel, it will be assumed that they are obtained by the extension process just described.

When the lattice F of step functions on a ring $\mathfrak{R}$ is replaced by K the definition of processes of additive type can be modified accordingly. However the notion of decomposable processes and the arguments leading to formula (11) needs further comment.

A simple way of reducing the problem to the previous case is the following. Suppose that φ defined on K is the characteristic function of a process of additive type. Suppose further that if a sequence $\{u_n\}$ of elements of K converges pointwise to zero and remains bounded by a fixed element of K, then $\varphi(u_n)$ converges to unity. Then, by the completion procedure described at the beginning of this section, the domain of φ can be extended to all Baire functions of the first class which are bounded and have compact support on S. We shall call the process decomposable if the extended process so obtained is decomposable.

4. Operations on positive random measures

In the present section S will be a locally compact space with countable base.

For the applications S will ordinarily be the intersection of an open set with a closed set in a Euclidean space. For theoretical purposes the results could be extended to cover the case where S is a Borel set in a Polish space or other absolutely measurable spaces. Measures on S will be assumed to be defined by their values on the space K of continuous numerical functions with compact support.

Let μ be a positive measure on S. Such a measure can be subjected to transformations among which the following four occur quite frequently. The first operation is the subordination operation. The other three will be described in quasi-mechanical terms.

(a) *Subordination.* Let L be a Paul Lévy measure in the interval $\Xi = (0, \infty)$. Let $\mu \otimes L$ be the product of μ and L on $S \times \Xi$. This product determines a decomposable process X on K. The sample values of the process are positive σ-additive integrals, and the characteristic function of the process is given by

$$\varphi(u) = E\{\exp [iX(u)]\} = \exp \left\{ \iint (e^{iu(s)\xi} - 1)\mu(ds)L(d\xi) \right\}. \tag{29}$$

The Poisson process associated with μ corresponds to the particular case where L is entirely concentrated at the point $\xi = 1$.

In some cases one may want to make L depend on the point $s \in S$. Instead of a product $\mu \otimes L$ one must then consider a measure on H as in theorem 1. The other three operations are easily described if μ is a mass carried by a single point s of S.

(b) *Deterministic transfer.* Let f be a map from S into itself (or more generally into another space S'). The mass μ situated at s is transferred to the point $f(s)$. If μ is a general measure having an image through this operation, said image will often be denoted $f(\mu)$.

(c) *Random transfer.* A mass μ situated at s is replaced by a mass μ situated

at a point t which is chosen at random according to some probability measure $T(s, dt)$. Distinct points are transferred independently of one another.

(d) *Smoothing or spreading.* A mass μ situated at s is replaced by an equal mass spread on S in such a way that the mass of a set A is $\mu T(s, A)$.

Note that the operations (a), (c), and (d) just described are essentially different. In particular (c) and (d) are entirely different. On the contrary (b) is a particular case of (d).

In operation (a) the only requirement to be imposed on L is that L be positive, carried by $(0, \infty)$ and such that both $\int \xi(1 + \xi^2)^{-1}L(d\xi)$ and $\int \xi^2(1 + \xi^2)^{-1}L(d\xi)$ be finite. In operations (b), (c), and (d) care has to be taken that the transformation does not lead to an infinite pileup within a compact. Thus we shall ordinarily require that f and T in (b) and in (c), (d), respectively, be proper Baire functions. That is, f is a Baire function for which the inverse images of compacts are relatively compact. Similarly, for every $u \in K$ the integral $\int u(t)T(s, dt)$ is a Baire function of s having compact support. In many cases, homogeneity or stationarity restrictions on the measures μ permit relaxation of these conditions.

The spreading operation described by a kernel T gives rise to a linear operation, again called T, which maps the space K onto the space B of bounded Baire functions with compact support. If φ is the characteristic function of a random measure X the characteristic function ψ of the process spread by T is simply

$$\psi(u) = \varphi(Tu) \tag{30}$$

for $u \in K$ and for the extension by sequential continuity of φ to B.

Similarly, if X is a positive random measure and if for $u \in K^- + iK$ one has $\varphi(u) = E \exp [X(u)]$, then the process Y subordinated to X and L has characteristic function

$$\psi(u) = E \exp [iY(u)] = \varphi\left[\int (e^{iu(s)\xi} - 1)L(d\xi)\right]. \tag{31}$$

What may correspond for general measures to the random transfer operation (c) is harder to see. However one can argue as follows. Suppose that a measure μ consists of a large number N of small masses $\{\delta_j\}$ carried by points $\{s_j\}$. After transferring each of the points independently of the others one would have masses δ_j at points t_j, and t_j would have distribution $T(s_j, dt_j)$. The Laplace transform of the resulting measure is, for $u \in K^-$, equal to

$$\varphi(u) = E \exp [\sum \delta_j u(t_j)] = \prod_j \int \exp [\delta_j u(t_j)]T(s_j, dt_j). \tag{32}$$

Under natural conditions, a passage to the limit for N increasing indefinitely and δ_j tending to zero shows that the point masses of μ would be transferred randomly but that the diffuse part of μ is transferred deterministically by the spreading operation T. This indicates that operations (a), (b), (c), and (d) do not represent accurately what happens, for instance, in the motion of a turbulent

fluid, the reason for this being that interactions of particles are ignored. To obtain a more reasonable representation of turbulent motion by linear operations it becomes necessary to work in the phase space of the fluid.

Equations (30) and (31) will be used repeatedly in the next section. However, since only positive random variables will be considered, it will be convenient to use instead of Fourier transforms the corresponding Laplace transforms defined on K^- only. In this case the subordination formula (31) becomes simply the following. If X has Laplace transform φ and Y is subordinated to X and L then the Laplace transform of Y is given by

$$\psi(u) = E \exp [Y(u)] = \varphi \left[\int (e^{u(s)\xi} - 1) L(d\xi) \right] \tag{33}$$

with $u \in K^-$ and $\varphi(u) = E \exp [X(u)]$.

Finally, note that since the Fourier transform can be extended to the space B of bounded Baire functions with compact support, the domain of a Laplace transform, such as ψ in (33), can be enlarged to B^-.

5. Formal description of the precipitation model

Let W be a locally compact subset of a plane or of the surface of a sphere. Consider a space S which is the product $S = W \times R \times V$ of W by a real line and an auxiliary locally compact space with countable base V.

A point $s = (w, t, v)$ of S will receive the following interpretation. The geographical location of s is indicated by w. The time of the event is given by t. The set V is used as a parameter set to indicate what kind of event occurred at w at time t.

In the following, μ will be a fixed positive measure on S. For $j = 1, 2, 3$, the letter T_j will denote a smoothing or spreading operation from S to S. Explicitly, let B denote the space of bounded Baire functions having compact support on S. The map T_j is a positive linear map from B to itself. The value $T_j u$ for $u \in B$ will also be denoted

$$[T_j u](s) = \int u(\sigma) T_j(s, d\sigma). \tag{34}$$

For reasons of convenience, it will not be assumed that T_j is normalized. However, it will be assumed that the values $T_j u$ for $u \in B$ are obtained by extension of the domain of T_j from K to B as explained in sections 3 and 4.

For $j = 1, 2, 3$ the letter L_j will denote a function assigning to each $s \in S$ a Paul Lévy measure $L_j(s, d\xi)$ on the interval $(0, \infty)$. It will be assumed that L_j corresponds to positive variables, that is, $L_j(s, d\xi)$ is a positive measure and

$$\int \frac{\xi + \xi^2}{1 + \xi^2} L_j(s, d\xi) < \infty. \tag{35}$$

Furthermore, it will be assumed that $s \to L_j(s)$ is a Baire function on S and that the integral (35) remains bounded whenever s stays in a compact of S.

The process describing rainfall is a linear process X on the space K (or B by extension) obtained as follows. Let X_1 be the process subordinated to μ and L_1. Let $X_2 = X_1T_1$ be the process obtained by smoothing X_1 through T_1. Let X_3 be subordinated to X_2 and L_2. Let $X_4 = X_3T_2$ be X_3 smoothed through T_2. Finally, let X_5 be subordinated to X_4 and L_3 and let $X = X_6 = X_5T_3$ be X_5 smoothed through T_3.

The interpretation of the processes X_j is the following. The measure μ describes general climatological conditions. For a set $A \subset S$, the measure $\mu(A)$ is proportional to the expected number of storms "centered" in A. Since μ is nonrandom, the process X_1 is a decomposable process. When L_1 has total mass equal to unity one can describe X_1 as follows. Points $\{s_k; k = 1, 2, \cdots,\}$ are chosen in S according to a Poisson process, the number of points falling in A having expectation $\mu(A)$. Each point s_k is then attributed a positive mass λ_k independently of everything else and according to the probability measure L_1. The chosen chosen points s_k are considered centers of storms. The strength or energy of the storm centered at s_k is proportional to λ_k.

The spreading operation T_1 is used to give storm extent and velocity. A particular smoothing kernel used by the author in preliminary studies can be used to describe the effect of T_1. If a point $s = (w, t, v)$ is chosen by the process X_1, then the point (w, t) in three-dimensional space is taken as the center of an ellipsoid. The directions of the axes of the ellipsoid depend on v and so do their magnitudes. The total mass attributed to the ellipsoid is equal to $\lambda T_1(s, S)$ if λ is chosen as explained above.

A section of the ellipsoid at time t gives an ellipse which represents a geographical area of instability. As t increases this elliptical area moves, the direction and velocity of motion being determined by the direction of the axes of the ellipsoid. The total mass attributed to the ellipsoid is spread uniformly over its volume.

The process X_2 is simply the superposition of all the ellipsoidal masses so obtained. It describes "storms."

The process X_3 is constructed from X_2 as X_1 was from μ. If L_2 is of total mass unity the description of X_3 could be given in terms quite similar to the above. The interpretation used will be the following. Given a sample measure m of X_2 one selects points to be centers of fronts according to the Poisson process defined by m. These functions are then attributed extent and velocity by the smoothing operation T_2. The strength of the front is selected according to L_2.

The interpretation of X_5 and X is now clear: given a sample function of X_4, that is, given the position and energy of the fronts one selects at random, Poissonwise, centers of convective cells in the area covered by the front. These cells are then given extent and motion through T_3. Finally, the amount of water precipitated by a cell is selected according to L_3.

The introduction of the set V gives to the model a very great degree of arbitrariness. This arbitrariness is further enhanced by the arbitrariness in the total mass of T_j and L_j. Indeed so much redundancy is introduced in this fashion that in the case where L_j is of finite total mass no generality is lost by concentrating

L_j at unity. To put it in different words, the elements T_j and L_j of the model are not necessarily identifiable. We shall indicate later why this redundancy was introduced in the model. However, before proceeding to specification problems let us mention that the Laplace transform of the process X can be written immediately following the rules established in sections 3 and 4.

Let u be a function belonging to the negative cone B^- of the space of Baire functions with compact support on S. For $u \in B$ let $T_j u$ and $Q_j u$ be the transforms obtained by the formulas

$$[T_j u](s) = \int u(\sigma) T_j(s, d\sigma), \tag{36}$$

$$[Q_j u](s) = \int [e^{u(s)\xi} - 1] L_j(s, d\xi). \tag{37}$$

Under the conditions imposed on T_j and L_j the images $T_j u$ and $Q_j u$ are also elements of B^-. Note that T_j is linear but that Q_j is definitely nonlinear. According to formulas (20) to (23) the Laplace transforms $\varphi_j(u) = E\{\exp [X_j(u)]\}$ can be computed recursively as follows.

$$\varphi_1(u) = \exp\left\{\int [Q_1 u]\mu(ds)\right\}, \tag{38}$$

$$\varphi_2(u) = \varphi_1(T_1 u), \tag{39}$$

$$\varphi_3(u) = \varphi_2(Q_2 u), \tag{40}$$

$$\varphi_4(u) = \varphi_3(T_2 u), \tag{41}$$

$$\varphi_5(u) = \varphi_4(Q_3 u), \tag{42}$$

$$\varphi_6(u) = \varphi_5(T_3 u). \tag{43}$$

In other words, the Laplace transform of the process $X = X_6$ can be written

$$\varphi(u) = \exp\left\{\int [Q_1 T_1 Q_2 T_2 Q_3 T_3 u]\mu(ds)\right\}, \tag{44}$$

the operations Q and T being performed from right to left, as usual, in the order indicated.

The above formulas were originally developed to obtain a Laplace transform for stream flow. Once rainfall has reached the ground, it is further subjected to complicated processes of evaporation, infiltration, and runoff. Since our present object is not a discussion of the processes, let us assume for convenience that the amount of water subject to surface runoff is proportional to the amount reaching the ground. The immediate subsurface flow could also be included in the following considerations.

Surface runoff of the water is controlled by the Navier-Stokes equations. Since these equations are nonlinear it is difficult to introduce them directly in the construction of a Laplace transform. Fortunately for our purposes, a linear method called the unit hydrograph method was introduced into hydrology by L. K. Sherman in 1932 [8] (see [9] for some more recent references). Reduced to its simplest expression the method consists simply in assuming that the intensity

$X_7(\tau)$ of runoff at time τ produced at a particular point of a stream by the rain X can be written

$$X_7(\tau) = \int T_4(\tau, s)X(ds), \tag{45}$$

where T_4 is still another smoothing kernel. The Laplace transform of X_7 may then be obtained directly from (44).

An interpretation of formula (45) can be given in the following manner. The value $T_4(\tau, s)$ is the density of probability that a "molecule" of water falling at $s = (w, t, v)$ will arrive at the measuring station at time τ. If the watershed considered is small and rough and if the rain is not excessive, runoff occurs in small rivulets in a manner fairly analogous to the motion of a ball on a pinball machine. The various particles of water do not have too much chance of interaction. Since they are very numerous, the circumstances are fairly well described by operation (c), section 4, applied to a diffuse measure. According to the elaboration given there, this is simply a smoothing process of the type illustrated by (45).

For large watersheds, where the runoff has had time to collect itself in sizable streams, interactions become important and the argument is no longer valid. It is well known, however, that the unit hydrograph method cannot be applied directly to large watersheds.

Up to now we have taken advantage of the linear structure of the transformations involved to work only with Laplace transforms. It should be pointed out that the Laplace transform of X can be used to derive the distribution of the number of rainy days in a week or a month, even though the passage from X to these numbers is not linear. A rather cumbersome but effective procedure can be described as follows.

Let A be a subset of S partitioned into n disjoint sets $\{A_j; j = 1, 2, \cdots, n\}$. Let $X^j = X(I_{A_j})$. The joint Laplace transform of the X^j can be read from $\varphi(u)$, with $u = \sum \alpha_j I_{A_j}$ and $\alpha_j \leqq 0$.

Consider a sequence $\epsilon = \{\epsilon_j;\ j = 1, 2, \cdots, n\}$ where ϵ_j is equal either to zero or to unity and take the product

$$p(\epsilon, \alpha) = \prod_{j=1}^{n} \{\epsilon_j \exp(\alpha X^j) + (1 - \epsilon_j)[1 - \exp(\alpha X^j)]\}. \tag{46}$$

The expectation of $p(\epsilon, \alpha)$ can be expressed in terms of φ. Furthermore, if α tends to $(-\infty)$ this expectation converges to the probability that the A_j for which $\epsilon_j = 1$ receive positive rain while the A_j for which $\epsilon_j = 0$ receive no rain at all. The generating function of the number of A_j receiving rain can be obtained by taking the expectation of

$$\sum_{\epsilon} \prod_{j=1}^{n} z^{\epsilon_j}\{\epsilon_j \exp(\alpha X^j) + (1 - \epsilon_j)[1 - \exp(\alpha X^j)]\} = \prod_{j=1}^{n} \{z + (1 - z)\exp(\alpha X^j)\} \tag{47}$$

and letting α tend to $-\infty$.

Joint Laplace transforms for numbers of rainy sets and total amount of rain can be obtained similarly. Even for moderately large values of n the expressions so obtained become rapidly very complicated. However, for a week divided into days, that is, for $n = 7$, the work involved is not excessive.

To conclude, let us give some indication of the problems of specification of the model. To simplify, let us first make the obviously inadequate assumption that the shower cells have sizes, velocities, and strength which are affected only by the geographical location and time of occurrence of their centers. Make similar assumptions on the characteristics of fronts and storms, so that, for instance, the energy of a front is independent of the energy of the storm where it occurs. Under these conditions one can dispense with V altogether and take for T_j normalized smoothing kernels. The set S is simply $S = W \times R$.

Let $A = A_1 \times A_2$ be a subset of $W \times R$. The measure $\mu(A)$ of A must reflect the yearly periodicity of meteorological phenomena so that $\mu(A)$ depends on the time of the year covered by A_2. Similarly, some parts of the world are more prone to instability than others. A rough indication of the variations of μ could be obtained through counting the cloudy days in the area A_1 for a long period of time.

Indications on T_1 can be obtained by a study of the duration, geographical extent, and prevailing direction of storms.

If our ficticious cell systems did actually correspond to natural fronts, indications on L_1 and T_2 could be obtained through a study of the number, extent, and motion of fronts in storms. Otherwise one would obtain some information simply by counting the number of breaks in the storms and following their geographical travels.

Another type of information on the T_j and L_j is supplied by the computation of correlations between amounts of rainfall in different periods and different locations. The theoretical formulas for such correlations can easily be derived from (44). Correlations between numbers of rainy days and total amount of rain, which can be obtained from formulas of the type (46), (47) provide a different type of information. For instance, the regression of the total amount of rain in a week on the number of rainy days in this week, which would be linear for a pure Poisson process, is here a complicated convex function in qualitative agreement with the observation. Finally, the actual distribution of rainfall and number of rainy days can also be taken into account.

The great arbitrariness inherent in the choice of the operators Q and L will at first prevent the application of any but the crudest estimation procedures. However, the moments of the distribution of rainfall can easily be derived from (44). Since they depend only on certain integrals involving T_j and L_j these integrals may be estimated.

Once enough information is obtained through the use of such crude methods one may attempt to give T_j and L_j parametric forms and use more reasonable estimation procedures.

The main comment to be made at this point on the applicability of purely

statistical methods is that they should not be applied to X alone, but, in the event that the system of storms, fronts, and cells used to describe X would correspond to some actual meteorological events, the joint distribution of the processes X_2, X_4, X_6 should be used.

Observed joint distributions of numbers of rainy days and total amounts of rain in two successive short periods at the same location computed by this author do not seem to require for their explanation any dependence of the amount of rain in a shower on the other characteristics of fronts and storms. However, this should not hold true in general. For instance, the amount of rain in a shower should depend on the type of air mass in which the shower cell occurs. To take such dependence into account the introduction of the auxiliary set V becomes necessary.

The introduction of the set V is also necessary to take proper care of the effect of geographical accidents such as mountain ranges. An *ad hoc* procedure, which has the merit of simplicity if not of accuracy, would be to avoid the introduction of V and simply give to mountain ranges larger and more numerous showers.

More numerous showers can be obtained by making L_2 depend on geographical location. Larger showers can be obtained by making L_3 depend on location. For instance, by varying L_3 only, we can pass from a few enormous showers to a continuous drizzle as explained in section 2. Modifying L_2 in a similar manner one can pass from geographically isolated cells to a continuum of instability in which the smoothing afforded by T_3 appears rather unnecessary.

Similar arguments apply to the seasonal variations of the characteristics of rainfall. By making L_3 depend on time one can obtain spring showers and winter drizzles.

For the effect of mountain ranges, a more realistic procedure would be to introduce in V a parameter describing the direction and velocity of the masses of air which carry the fronts and make L_3 depend on the relative orientation of the range with respect to the motion of the air. One can also make the velocity of the "front" depend on the location and orientation.

The main difficulty in such circumstances is that, in a model of this complexity, it becomes more and more difficult to estimate or test anything through purely statistical methods. It is then necessary to specify some of the elements of the model through purely physical arguments. We hope to present, in the near future, a modification of the model in which this will be possible.

The basic features of the model described in the present paper were elaborated while the author was a statistician at Electricité de France in Paris, where we had numerous and lengthy discussions with G. Morlat. While attempting to relate the model to meteorological phenomena we had the benefit of advice and criticism of Dr. A. Court, to whom we would like to extend our heartiest thanks.

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